LINEAR ALGEBRA, MMATH END SEMESTRAL EXAMINATION, DECEMBER 2009

Please attempt all questions. If you use any theorem which has been taught in class in your answer, then please state it in full and clearly. All questions carry equal marks.

Question 1 Let $A \in O_2(\mathbb{R})$ be an orthogonal 2×2 real matrix. Suppose $det(A) = -1$. Prove that A represents a reflection of the plane about a line through the origin.

Question 2 Let $V = M_2(\mathbb{R})$ be the real vector space of all real 2×2 matrices. Define a symmetric bilinear form $<,>$ on V by $< A, B >=$trace(AB) for $A, B \in V$. Find the signature of this form.

Question 3 Suppose that $A, B \in M_n(\mathbb{C})$ are two hermitian and positive definite matrices. Which of the following matrices are hermitian and positive definite: $A^2, A^{-1}, AB, A+B$. In each case, give an argument if your answer is yes, or, give a counterexample if your answer is no.

Question 4 Let $A \in M_n(F)$ be a $n \times n$ $(n \geq 2)$ nilpotent matrix over the field F. Prove that $A^n = 0$. Now suppose that $A^{n-1} \neq 0$. Prove that A does not have a square root, i.e. there does not exist a matrix $B \in M_n(F)$ such that $B^2 = A$.

Question 5 Let V be a finite dimensional vector space over a field F and let $T \in End_F(V)$. Consider V as an $F[x]$-module via the action $g(x).v := g(T)(v)$ for all $v \in V$ and $g(x) \in F[x]$. Prove that the characteristic polynomial of T is an irreducible polynomial over F, if and only if, every non-zero vector in V generates V as an $F[x]$-module.

Question 6 Let $A \in M_n(\mathbb{C})$ be a $n \times n$ complex matrix, and let A^t be the transpose of A. Prove or disprove: A is similar to A^t over $\mathbb{C}$.